International Journal of Theoretical Physics, Vol. 44, No. 4, April 2005 (© 2005)
DOI: 10.1007/s10773-005-3980-y

Application of Entangled State Representation to Deriving Normally Ordered Expansion of 1-Dimensional Coulomb Potential

Hong-yi Fan[1,2,4] and Liang Fu[3]

Received August 2, 2004; accepted August 26, 2004

Guided by Dirac's advice that "When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible," we construct the entangled state representation to derive the normally ordered expansion formula of the 1-dimensional two-body Coloumb potential. The method of integration within an ordered product of operators is also used. Further application of the new formula in some perturbation calculation is discussed.

KEY WORDS: bipartite entangled state; 1-dimensional bipartite Coulomb potential; IWOP method.

1. INTRODUCTION

In recent years, quantum entanglement (Einstein *et al.*, 1935) and entangled states have received much attention of physicists because they play an essential role in quantum communication and quantum computation (Bennett *et al.*, 1895; Ekert and Josza, 1996; DiVincenzo, 1995; Huches *et al.*, 1997; Braunstein and Kimble, 1998). In this work we show that the entangled state representations can be applied to solving some two-body operator ordering problems. Operator re-ordering (normal ordering, antinormal ordering, and Weyl ordering) is frequently encountered in all fields relating to quantum mechanics. The normally ordered expansions of operators are very helpful in calculating their coherent state

[1] Department of Material Science and Engineering, University of Science, and Technology of China, Hefei, Anhui, 230026, China.
[2] Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, China.
[3] Special Class for the Gifted Young, University of Science and Technology, of China, Hefei, Anhui, 230026.
[4] To whom correspondence should be addressed at Department of Material Science and Engineering, University of Science, and Technology of China, Hefei, Anhui, 230026, China.

expectation values (Gl‘auber, 1963; Klauder and Skargerstam, 1985) because

$$ {}_1\langle z'| : f(a_1^\dagger, a_1) : |z\rangle_1 = f(z'^*, z)\langle z'|z\rangle, \tag{1} $$

where the symbol : : denotes normal ordering, $|z\rangle$ is the coherent state,

$$ |z\rangle_1 = \exp\left[-\frac{1}{2}|z|^2 + za_1^\dagger\right]|0\rangle_1, \quad a_1|z\rangle_1 = z|z\rangle_1. \tag{2} $$

In Ref. (Louisell, 1973) Louisell devoted a whole chapter to studying operator ordering problems. The approach he took is mainly via the coherent state representation. In Refs. (Hong-yi Fan *et al.*, 1987; Hong-yi, 2003; Wünsche, 1999) the method of integration within an ordered product (IWOP) of operators is introduced by which one can easily derive the normal product form of many operators. For instance, recasting the completeness relation of coherent state into normal ordering,

$$ \int \frac{d^2z}{\pi}|z\rangle_{11}\langle z| = \int \frac{d^2z}{\pi} : e^{-|z|^2+za_1^\dagger+z^*a_1-a_1^\dagger a_1} := 1, \tag{3} $$

and using the mathematical formula

$$ \int \frac{d^2z}{\pi} z^n z^{*m} e^{A|z|^2+Bz+Cz^*} = e^{-BC/A}\sum_{l=0} \frac{n!m!}{l!(n-l)!(m-l)!(-A)^{n+m-l+1}} B^{m-l}C^{n-l}, \quad ReA < 0, \tag{4} $$

we can directly put $a_1^n a_1^{\dagger m}$ into normal ordering,

$$ a_1^n a_1^{\dagger m} = \int \frac{d^2z}{\pi} z^n z^{*m}|z\rangle\langle z| = \int \frac{d^2z}{\pi} z^n z^{*m} : e^{-|z|^2+za_1^\dagger+z^*a_1-a_1^\dagger a_1} : $$
$$ = \sum_{l=0}^{\min(m,n)} : \frac{n!m!}{l!(n-l)!(m-l)!} a_1^{\dagger m-l} a_1^{n-l} : . \tag{5} $$

For another example, using the normally ordered form of completeness relation of coordinate eigenstates (Hong-yi, 2003)

$$ \int_{-\infty}^{\infty} \mathrm{d}x|x\rangle_{11}\langle x| = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} : \exp[-(x-\hat{X}_1)^2] := 1 \tag{6} $$

where $\hat{X}_1|x\rangle_1 = x|x\rangle_1$, as well as the IWOP technique, we have obtain for $n > 0$ integer,

$$ \hat{X}_1^n = \int_{-\infty}^{\infty} dx\, \mathrm{d}x\, x^n|x\rangle_{11}\langle x| = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} x^n : \exp\left[-(x-\hat{X}_1)^2\right] : $$

$$= \frac{1}{\sqrt{\pi}} : \sum_{l=0}^{[n/2]} \Gamma\left(l + \frac{1}{2}\right) \binom{n}{2l} \hat{X}_1^{n-2l} :, \tag{7}$$

where Γ is the Gamma function; and (Hong-yi and Liang, 2003)

$$\hat{X}_1^{-n} = \sqrt{\pi}(-1)^n : \sum_{m=\frac{[n-1]}{2}}^{\infty} \frac{(-1)^m}{\Gamma(m+1/2)} \binom{2m-1}{n-1} \hat{X}_1^{2m-n} : . \tag{8}$$

While the normally ordered expansion of single partite operators has been widely studied, there is less attention on the case of bipartite operators. However, bipartite states can exhibit more interesting behaviors than their single-partite counterparts, such as quantum entanglement, which is widely explored in quantum information. Therefore, it is natural and justified to extend our attention to bipartite operators and study their normally ordered expansions, which may be of potential uses. Coulomb potential between two charged particles plays a dominant role in many quantum mechanical dynamic problems. An interesting question thus naturally arises: what is the normally ordered expansion of one-dimensional two-body Coulomb potential $(X_1 - X_2)^{-1}$? Here $X_i = \frac{1}{\sqrt{2}}(a_i^\dagger + a_i)$ is the coordinate operator, $[a_i, a_j^\dagger] = \delta_{i,j}$, $i, j = 1, 2$.

One may naturally think of the two-mode coordinate repesentation $|x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$, and use its completeness relation to write $(X_1 - X_2)^{-1}$ as

$$(X_1 - X_2)^{-1} = \int\int_{-\infty}^{\infty} dx_1\, dx_2 \frac{1}{x_1 - x_2} |x_1, x_2\rangle\langle x_1, x_2| \tag{9}$$

and then use the IWOP method to perform the following integration

$$(X_1 - X_2)^{-1} = \frac{1}{\pi} \int\int_{-\infty}^{\infty} dx_1\, dx_2 \frac{1}{x_1 - x_2} : \exp\left[-(x_1 - \hat{X}_1)^2 - (x_2 - \hat{X}_2)^2\right] : . \tag{10}$$

However, the integral is not separable into two independent integrals over dx_1 and dx_2 respectively, moreover, when $x_1 = x_2$, there arises singularity. To avoid such difficulty, instead of using the direct product of two single-mode coordinate repesentation $|x_1, x_2\rangle$, we shall find a suitable representaion for diagonlizing $(X_1 - X_2)^{-1}$, this will greatly save our labouring. As Dirac's guided in (Dirac, 1958): "When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a represenation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible," we find that by employing the entangled state representation $|\eta\rangle$, (Hong-yi and Klauder, 1994; Hong-yi and Yue, 1996; Hongyi, 2002) which is the common eigenvector of $X_1 - X_2$ and $P_1 + P_2$, to tackle this problem. The quantum entanglement involved in a bipartite's relative coordinate $X_1 - X_2$ and the total momentum $P_1 + P_2$ was first discussed by Einstein-Podolsky-Rosen (EPR) in their argument that the quantum mechanics is incomplete (Einstein *et al.*, 1935), constructing the corresponding entangled state representation can express this kind of entanglement in a transparent fashion. The work is arranged as follows: In Sec. 2 we briefly list the main properties, including its Schimidt decomposition, of $|\eta\rangle$. In Sec. 3 we deduce the normally ordered expansion formula of $(X_1 - X_2)^{-1}$. In Sec. 4 we further deduce the normally ordered expansion formula of $(X_1 - X_2)^{-n}$. In Sec. 5 we apply the new formula in some pertubation calculations.

2. THE BIPARTITE ENTANGLED STATE REPRESENTATION

In Ref. (Hong-yi and Klauder, 1994) the explicit form of the common eigenvectors $|\eta\rangle$ of two particles' relative position $X = X_1 - X_2$ and the total momentum $P = P_1 + P_2$ are constructed in the two-mode Fock space

$$|\eta\rangle = \exp\left[-\frac{|\eta|^2}{2} + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{11}$$

where $\eta = \eta_1 + \eta_2$ is a complex number. It is remarkable that η's real part and imaginary part are respectively the eigenvalues of $X_1 - X_2$ and $P_1 + P_2$, i. e.,

$$\frac{1}{\sqrt{2}}(X_1 - X_2)|\eta\rangle = \eta_1\,|\eta\rangle\,,\ \frac{1}{\sqrt{2}}(P_1 + P_2)|\eta\rangle = \eta_2|\eta\rangle, \tag{12}$$

where $P_i = \frac{1}{\sqrt{2}\mathrm{i}}(a_i - a_i^\dagger)$. Using the IWOP method we can neatly prove that $|\eta\rangle$ states span an complete set

$$\int \frac{d^2\eta}{\pi}\,|\eta\rangle\,\langle\eta| = \int \frac{d^2\eta}{\pi} : \exp\{-[\eta - (a_1 - a_2^\dagger)][\eta^* - (a_1^\dagger - a_2)]\} := 1,$$
$$d^2\eta = d\eta_1 d\eta_2, \tag{13}$$

or

$$\int \frac{d^2\eta}{\pi}\,|\eta\rangle\,\langle\eta| = \int \frac{d^2\eta}{\pi} : \exp\left\{\left[-\eta_1 - \frac{1}{\sqrt{2}}(X_1 - X_2)\right]^2 - \left[\eta_2 \frac{1}{\sqrt{2}}\,(P_1 + P_2)\right]^2\right\} := 1, \tag{14}$$

which bears some formal correspondence to (12) and thus is easily remembered. Note that $|\eta\rangle$ are orthonormal to each other

$$\langle\eta \mid \eta'\rangle = \pi\delta^{(2)}(\eta - \eta') = \pi\delta(\eta_1 - \eta_1')\delta(\eta_2 - \eta_2'). \tag{15}$$

Thus $|\eta\rangle$ is qualified to be a quantum mechanical representation. The Schimidt decomposition of $|\eta\rangle$ in the coordinate eigenvector space is (Hong-yi, 2002)

$$|\eta\rangle = e^{-i\eta_2\eta_1}\int_{-\infty}^{\infty} \mathrm{d}x|x\rangle_1 \otimes |x-\sqrt{2}\eta_1\rangle_2 e^{i\sqrt{2}\eta_2 x}, \tag{16}$$

and in mometum eigenvector space is

$$|\eta\rangle = e^{i\eta_1\eta_2}\int_{-\infty}^{\infty} dp|p\rangle_1 \otimes |\sqrt{2}\eta_2 - p\rangle_2 e^{-i\sqrt{2}\eta_1 p}. \tag{17}$$

Using (14) and (12) we can write $\frac{1}{X_1-X_2}$ as

$$\frac{1}{X_1-X_2} = \int \frac{d^2\eta}{\pi}\frac{1}{X_1-X_2}|\eta\rangle\langle\eta| = \int \frac{d^2\eta}{\pi} : \frac{1}{\eta_1}\exp\left\{\left[-\eta_1 - \frac{1}{\sqrt{2}}(X_1-X_2)\right]^2 - \left[\eta_2 - \frac{1}{\sqrt{2}}(P_1+P_2)\right]^2\right\} : . \tag{18}$$

Now we see the advantage of using $|\eta\rangle$, that is the integral (18) is saparable into two-independent integrals over $d\eta_1$ and $d\eta_2$. Integrating over η_2 we have

$$\frac{1}{X_1-X_2} =: \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\eta_1 \frac{1}{\eta_1}\exp\left\{\left[-\eta_1 - \frac{1}{\sqrt{2}}(X_1-X_2)\right]^2 :=: \frac{1}{\sqrt{2\pi}}\int_0^{\infty} d\eta_1 e^{-\eta_1^2} \times \frac{\exp[\sqrt{2}\eta_1(X_1-X_2)] - \exp[-\sqrt{2}\eta_1(X_1-X_2)]}{\eta_1}\exp[A] :, \tag{19}$$

where $A \equiv -\frac{(X_1-X_2)^2}{2}$. (19) has a singualrity for $\eta_1 = 0$, so the integral over $d\eta_1$ should be done in the sense of principal-value integration

$$\frac{1}{X_1-X_2} =: \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\int_0^{\infty} d\eta_1 e^{-\eta_1^2}\frac{2^{k+\frac{3}{2}}\eta_1^{2k}}{(2k+1)!}(X_1-X_2)^{2k+1}\exp[A] : \tag{20a}$$

$$=: \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2})}{(2k+1)!}(X_1-X_2)^{2k+1}\exp[A] : \tag{20b}$$

$$=: \sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k+1)!}(X_1-X_2)^{2k+1}\exp[A] : \tag{20c}$$

$$=: \sum_{k=0}^{\infty}\frac{1}{(2k+1)2^k k!}(X_1-X_2)^{2k+1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(X_1-X_2)^{2m}}{2^m} : \tag{20d}$$

$$=: \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(2k+1)2^n k!(n-k)!}(X_1-X_2)^{2n+1} : \tag{20e}$$

$$=:\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(2k+1)}\right)\frac{(-1)^n}{2^n n!}(X_1-X_2)^{2n+1}:. \tag{20f}$$

Using the combinatorial formula, (Gould, 1972)

$$\sum_{k=0}^{n}\frac{x}{k+x}(-1)^k\binom{n}{k}=\frac{1}{\binom{n+x}{n}}, \tag{21a}$$

(20) becomes

$$\frac{1}{X_1-X_2}=:\sum_{n=0}^{\infty}\frac{1}{\binom{n+\frac{1}{2}}{n}}\frac{(-1)^n}{2^n n!}(X_1-X_2)^{2n+1}:$$

$$=\sqrt{\pi}:\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)2^{n+1}}(X_1-X_2)^{2n+1}:, \tag{22a}$$

this is the normally ordered expansion of the 1-dimensional two-body Coulomb potential. We can check the above result by calculating

$$(X_1-X_2)\frac{1}{X_1-X_2}=\frac{1}{\sqrt{2}}(a_1^\dagger+a_1-a_2^\dagger+a_2):$$

$$\times\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1-X_2)^{2n+1}: \tag{23a}$$

Using the formula

$$a_i:f(a_i^\dagger,a_i):=:a_i f\left(a_i^\dagger,a_i\right):+:\frac{\partial f}{\partial a_i^\dagger}:,\quad i=1,2, \tag{24a}$$

we have

$$\frac{1}{\sqrt{2}}(a_i^\dagger+a_i):\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)2^{n+1}}(X_1-X_2)^{2n+1}:=\frac{1}{\sqrt{2}}:\sqrt{\pi}\sum_{n=0}^{\infty}$$

$$\times\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}a_i^\dagger(X_1-X_2)^{2n+1}:+\frac{1}{\sqrt{2}}:\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}a_i$$

$$\times(X_1-X_2)^{2n+1}:+\frac{1}{\sqrt{2}}\left[a_i,:\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)2^{n+1}}(X_1-X_2)^{2n+1}:\right]$$

$$=:\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}X_i(X_1-X_2)^{2n+1}:+\frac{1}{2}:\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{\Gamma(n+\frac{3}{2})2^{n+1}}$$

$$\times(X_1-X_2)^{2n}:. \tag{25a}$$

Substituting (25) into (23) we have

$$
\begin{aligned}
(X_1 - X_2)\frac{1}{X_1 - X_2} =&: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1 - X_2)^{2n+2} \\
&\times :+: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1 - X_2)^{2n} : \\
&=: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1 - X_2)^{2n+2} : +1 \\
&+ : \sqrt{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1 - X_2)^{2n} : \\
&=: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(X_1 - X_2)^{2n+2} : +1 \\
&+ : \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+3)}{\Gamma(n+\frac{5}{2})2^{n+2}}(X_1 - X_2)^{2n+2} := 1.
\end{aligned}
\tag{26a}
$$

3. NORMALLY ORDERED EXPANSION OF $(X_1 - X_2)^{-n}$

To derive the normally ordered expansion of $(X_1 - X_2)^{-n}$, we notice that there exists the mathematical formula (Gradshteyn and Ryzhik, 1980)

$$
\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}\exp[-(x-y)^2]H_n(x) = (2y)^n, \tag{27}
$$

where H_n denotes the n-th Hermite polynomials. It then follows from (12) and (14) that

$$
\begin{aligned}
H_n\left(\frac{X_1 - X_2}{\sqrt{2}}\right) &= \int\frac{d^2\eta}{\pi} : H_n(\eta_1)\exp\left\{\left[-\eta_1 - \frac{1}{\sqrt{2}}(X_1 - X_2)\right]^2\right. \\
&\left. -\left[\eta_2 - \frac{1}{\sqrt{2}}(P_1 + P_2)\right]^2\right\} := \int\frac{d\eta_1}{\sqrt{\pi}} : H_n(\eta_1) \\
&\times \exp\left\{\left[-\eta_1 - \frac{1}{\sqrt{2}}(X_1 - X_2)\right]^2\right\} := \sqrt{2^n} : (X_1 - X_2)^n :,
\end{aligned}
\tag{28}
$$

Comparing it with the well-known recurrence relation of the Hermite polynomials $H_n^{'}(x) = 2nH_{n-1}(x)$, we see the differentiation rule about the normal ordering

$$\begin{aligned}\frac{d}{dX_1} &: (X_1 - X_2)^n := \sqrt{2^{-n}}\frac{d}{dX_1}H_n\left(\frac{X_1 - X_2}{\sqrt{2}}\right) \\ &= n\sqrt{2^{1-n}}H_{n-1}\left(\frac{X_1 - X_2}{\sqrt{2}}\right) = n : (X_1 - X_2)^{n-1} :=: \frac{d}{dX_1}(X_1 - X_2)^n :,\end{aligned} \tag{29}$$

which means $\frac{d}{dX_1}$ can operate across the border : : of : $(X_1 - X_2)^n$:. It then follows

$$\begin{aligned}\frac{1}{(X_1 - X_2)^n} &= \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{d\hat{X}_1}\right)^{n-1}\frac{1}{X_1 - X_2} \\ &= \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{d\hat{X}_1}\right)^{n-1} : \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+\frac{3}{2})2^{m+1}}(X_1 - X_2)^{2m+1} : \\ &= \frac{(-1)^{n-1}}{(n-1)!} : \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+\frac{3}{2})2^{m+1}}\left(\frac{d}{d\hat{X}_1}\right)^{n-1}(X_1 - X_2)^{2m+1} : \\ &= (-1)^{n-1} : \sqrt{\pi}\sum_{m=[\frac{n-1}{2}]}^{\infty}\frac{(-1)^m\binom{2m+1}{n-1}}{\Gamma(m+\frac{3}{2})2^{m+1}}(X_1 - X_2)^{2m-2n+2} : \\ &= (-1)^n : \sqrt{\pi}\sum_{m=[\frac{n+1}{2}]}^{\infty}\frac{(-1)^m\binom{2m-1}{n-1}}{\Gamma(m+\frac{1}{2})2^m}(X_1 - X_2)^{2m-2n} :,\end{aligned} \tag{30}$$

which is the normally ordered expansion of $(X_1 - X_2)^{-n}$. In particular, we have

$$\begin{aligned}\frac{1}{(X_1 - X_2)^2} &=: \sqrt{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m\binom{2m-1}{n-1}}{\Gamma\left(m+\frac{1}{2}\right)2^m}(X_1 - X_2)^{2m-2} : \\ &=: -\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)}{\Gamma\left(m+\frac{3}{2}\right)2^{m+1}}(X_1 - X_2)^{2m} : .\end{aligned} \tag{31}$$

Similarly, we can use (12) and (14) to derive the normally ordered expansion of $\frac{1}{P_1+P_2}$,

$$\frac{1}{P_1 + P_2} = \sqrt{\pi} : \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(P_1 + P_2)^{2n+1} : . \tag{32}$$

4. APPLICATION OF (22)

Using (22) the operation of $\frac{1}{X_1-X_2}$ on the vacuum state is

$$\frac{1}{X_1 - X_2}|00\rangle = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}}(a_1^\dagger - a_2^\dagger)^{2n+1}|00\rangle. \tag{33}$$

From the normally ordered expansion of $\frac{1}{X_1-X_2}$ we can immediately read off its coherent state matrix element of

$$\langle z_1', z_2'|\frac{1}{X_1 - X_2}|z_1, z_2\rangle = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})2^{n+1}} \times\left(\frac{z_1+z_1^{'*}}{\sqrt{2}} - \frac{z_2+z_2^{'*}}{\sqrt{2}}\right)^{2n+1} e^{z_1z_1^{\prime *}+z_2z_2^{\prime *}} \tag{34}$$

As an application of the above operator formula, we consider a system of two harmonic oscillators, with the Hamiltonian being $H_0 = a_1^\dagger a_1 + a_2^\dagger a_2$, supposing this system is initially in its ground states. Suddenly, the two oscillators become charged, which means a 1-dimensional Coulomb-like potential $\frac{g}{X_1-X_2}$ is abruptly exerted. If the coupling intensity g is small enough, a perturbation calculation for the ground state energy shift using (34) is

$$\langle 00|\frac{g}{X_1 - X_2}|00\rangle = \sqrt{\frac{2}{\pi}}g. \tag{35}$$

On the other hand, if the two oscillators originally rest on the coherent state $\left|z_1,\ z_2'\right\rangle$, and $g,\ |z_1|,\ \left|z_2'\right|$ are small enough, then from (34) we know that the perturbation results in (up to the second order of $|z_1|$ and $\left|z_2'\right|$)

$$\langle z_1,\ z_2'|\frac{g}{X_1 - X_2}|z_1,\ z_2'\rangle = \sqrt{\frac{2}{\pi}}g\left[1 - \frac{1}{12}\sum_{i=1}^{3}\left(z_{\alpha i} + z_{\alpha i}^* - z_{\beta i}' - z_{\beta i}^{\prime *}\right)^2\right]. \tag{36}$$

In summary, we have derived the normally ordered expansion formulas of Coulomb potential $\frac{1}{(X_1-X_2)}$ and $(X_1 - X_2)^{-n}$ by virtue of the method of integral within an ordered product of operators. It is the entangled state representation that brings the convenience and brevity for our calculation. We expect that the new expansion formula would have more applications besides in perturbation calculations.

ACKNOWLEDGMENT

Work supported by the President Foundation of Chinese Academy of Science and National Natural Science Foundation of China under grant 10175057.

REFERENCES

Bennett, C. H., *et al.* (1993). *Physical Review Letters* **70**, 1895.
Braunstein, S. L. and Kimble, H. J. (1998). *Physical Review Letters* **80**, 869.
Dirac, P. A. M. (1958). *Principles of Quantum Mechanics*, 3rd edn, Oxford University Press.
DiVincenzo, D. P. (1995). *Science* **270**, 255.
Einstein, A., Podolsky, B., and Rosen, N. (1935). *Physical Review* **47**, 777.
Ekert, A. and Josza, R. (1996). *Review of Modern Physics* **68**, 733.
Glauber R. J. (1963). *Physical Review* **130**, 1529.
Gould, H. W. (1972). Combinatorial Identities, Morgantown.
Gradshteyn, I. S. and Ryzhik, I. M. (1980). In Alan Jeffery (ed.), *Table of Integrals, Series and Products*, Academic Press, New York.
Hong-yi, F., Zaidi, H. R., and Klauder, J. R. (1987). *Physal Review D* **35**, 1831.
Hong-yi, F. and Klauder, J. R. (1994). *Physical Review A* **49**, 704.
Hong-yi, F. and Yue, F. (1996). *Physical Review A* **54**, 958.
Hong-yi, F. (2003). *Journal of Optics B: Quanum Semiclass Optics R* **5**, R141.
Hongyi, F. (2002). *Physical Review A* **65**, 064102.
Hong-yi, F. and Liang, F. (2003). *Journal of Physics A* **36**, 4987.
Hong-yi, F. (2002). *Physica Letters A* **294**, 253.
Huches, C. A., *et al.* (1997). *Physcal Review A* **56**, 1163, and references therein.
Klauder, J. R. and Skargerstam, B. S. (1985). Coherent States, World Scientific, and many references therein.
Louisell, W. H. (1973). Quantum Statistical Properties of Radiation, New York, John Wiley.
Wünsche, A. (1999). *Journal of Optics B: Qaunum Semiclass Optics R* **1**, 11.